Int. J. Solids Structures Vol. 29, No. 18, pp. 2259–2279, 1992
Printed in Great Britain.

0020–7683/92 $5.00+.00

THERMOELASTIC PROPERTIES OF COMPOSITES WITH SHORT COATED FIBERS

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(*Received* 27 *December* 1990; *in revised form* 20 *November* 1991)

Abstract—The effective thermoelastic properties of composites with coated short fibers are derived in this work. Under the assumption of thin coating, the stress field of the coated layer remains uniform across the thickness of the layer but otherwise possessing variation along other directions and can be found in terms of the stress field of the fiber and the direction cosines through the use of the interface jump condition between the coating and the fiber. The effective thermoelastic properties are then derived based on Mori-Tanaka's scheme and the modified Walpole method. Numerical comparisons to existing results for long coated fibers are made. In addition, a parametric study is also included.

INTRODUCTION

Composite materials have been extensively used in many applications, particularly as a structural component due to their high specific stiffnesses and strengths. In most applications the fibers are not coated. Recently, there has been a growing demand for coated fibers as a reinforcement in some new application areas such as electrical composites, metal matrix composites (MMC) and ceramic matrix composites (CMC) intended for high temperature applications. Improvement in the bonding between the fiber and the matrix, preventing oxidation of the fiber, and introducing transition properties are the basic functions of coating.

The basic problem in the composites with coated fibers is the calculation of the thermoelastic stress field and its properties. Walpole (1978) proposed a simple method to calculate the stress field within a thin coating if the solutions to the stress field are known for a single noncoated fiber embedded in an infinite matrix, avoiding actually solving the elastic field. Mikata and Taya (1985, 1986) applied Boussinesq–Sadowsky stress functions to calculate the stress field for two confocal prolate spheroids embedded in an infinite body. Hatta and Taya (1987) calculated the thermal stress field within the coating for a coated fiber composite by modifying the Walpole method together with the Eshelby equivalent inclusion method. The above mentioned works limited themselves to the stress field but not the thermoelastic properties.

Benveniste *et al.* (1989), derived the thermoelastic stress field and effective properties for the composites with long continuous fibers. Chen *et al.* (1990), extended the above work to cylindrical orthotropic fiber composites. Pagano and Tandon (1988, 1990) gave their predictions to the thermoelastic properties for multidirectional coated fiber composites, in which the fibers were continuous. All of these works hold for both thin and thick coating but limit themselves to composites with long continuous fibers, partially due to the difficulty in solving the stress field of short coated fiber composites.

Thus, since there are no existing works showing the thermoelastic properties for short coated fiber composites, it is intended in this work to derive the tensorial expressions for the prediction of their effective thermoelastic properties. It will be assumed that the coated layer is thin and hence the stress field within the coating is reasonably assumed to be uniform through the thickness but otherwise may vary along other directions. Hence, Hatta and Taya's work (1987) is followed in this work to calculate the thermoelastic properties. Numerical comparisons with other existing works have been made and the results are satisfied. Parametric studies are also conducted and the results are presented below.

EFFECTIVE ELASTIC MODULI

Consider an infinite elastic body (without inclusions, i.e. the matrix only) subjected to a uniform stress field σ_0. The uniform strain ε_0 is

$$\varepsilon_0 = \mathbf{C}^{0^{-1}} \cdot \sigma_0 \tag{1}$$

in which $\mathbf{C}^0$ is the elastic modulus tensor of the matrix (which is assumed to be isotropic). It is noted that in this work the bold face capital English letters and small Greek letters denote the fourth order and the second order tensors respectively. When there are ellipsoidal inclusions (coated fibers) present in the matrix, a perturbed stress field is induced and is denoted as $\tilde{\sigma}(x)$. The total stress field $\sigma(x)$ is now the sum of two stress fields: $\sigma_0 + \tilde{\sigma}(x)$. Define the volumetric average of the perturbed stress and strain fields of the matrix, $\bar{\sigma}$ and $\bar{\varepsilon}$ according to

$$\begin{aligned}\bar{\sigma} &= \langle \tilde{\sigma}(x) \rangle_{D-\Omega} = \frac{1}{V_{D-\Omega}} \int_{D-\Omega} \tilde{\sigma}(x)\, \mathrm{d}V \\ &= \mathbf{C}^0 \cdot \bar{\varepsilon}\end{aligned} \tag{2}$$

in which

$D, \Omega, D-\Omega$: domain of the whole elastic body (composite), all the coated fibers and the matrix respectively,
$V_{D-\Omega}$: volume of the matrix,
$\langle\ \rangle_{D-\Omega}$: volumetric average for the domain $D-\Omega$.

Thus the average stress field in the matrix is

$$\sigma_m = \sigma_0 + \bar{\sigma} = \mathbf{C}^0 \cdot (\varepsilon_0 + \bar{\varepsilon}). \tag{3}$$

Denote $\mathbf{C}^f$ and $\mathbf{C}^c$ as elastic modulus tensors of the fiber and the coating. Then by use of Eshelby's equivalent inclusion method (Eshelby, 1957; Mori and Tanaka, 1973), one has:

in the domain of a typical fiber, Ω_1,

$$\begin{aligned}\sigma^f = \mathbf{C}^f \cdot \varepsilon^f &= \mathbf{C}^f \cdot (\varepsilon_0 + \bar{\varepsilon} + \varepsilon^{11} + \varepsilon^{12}) \\ &= \mathbf{C}^m \cdot (\varepsilon_0 + \bar{\varepsilon} + \varepsilon^{11} + \varepsilon^{12} - \varepsilon^{*1})\end{aligned} \tag{4}$$

and, in the domain of the coating of a typical fiber, $\Omega_2 - \Omega_1$,

$$\begin{aligned}\sigma^c = \mathbf{C}^c \cdot \varepsilon^c &= \mathbf{C}^c \cdot (\varepsilon_0 + \bar{\varepsilon} + \varepsilon^{22} + \varepsilon^{21}) \\ &= \mathbf{C}^m \cdot (\varepsilon_0 + \bar{\varepsilon} + \varepsilon^{22} + \varepsilon^{21} - \varepsilon^{*2})\end{aligned} \tag{5}$$

in which ε^{*1} and ε^{*2} are "eigenstrains" (Mura, 1982) defined in Ω_1 and $\Omega_2 - \Omega_1$, respectively, and ε^{ij} is the disturbance of the strain field in the ith domain due to the existence of the jth phase with i(or j) = 1 (fiber) and 2 (coating). From Eshelby (1957), we know

$$\varepsilon^{11} = \mathbf{S} \cdot \varepsilon^{*1}, \tag{6}$$

where S is Eshelby's Tensor (Eshelby, 1957; Mura, 1982).

In the present model, the disturbance of the strain field in Ω_1 due to the coating is averaged over the fiber domain Ω_1. The average of the disturbed strain field is denoted as ε^{12}. Thus we have $\varepsilon^{12} = \langle \tilde{\varepsilon}^{12}(x) \rangle_1$. Then according to Hatta and Taya (1987), under the assumption of a thin coating layer, one has

$$\varepsilon^{12} = \frac{f_2}{f_1}\langle S^0\cdot\varepsilon^{*2}\rangle_2 = f_{12}\langle S^0\cdot\varepsilon^{*2}\rangle_2 \tag{7}$$

in which f_1 and f_2 denote the volume fraction of fiber and coating, respectively, and $\langle\ \rangle_1$ and $\langle\ \rangle_2$ denote the averaged quantity over Ω_1 and $\Omega_2-\Omega_1$, respectively, and further, the fourth rank tensor S_0 is related to the Eshelby tensor S and the surface directional cosine n_k of a fiber surface outward unit normal $\mathbf{n}$ as:

$$S^0_{klmn} = S_{klmn} - C^{m}_{pqmn}n_q n_l K^{m^{-1}}_{kp} \tag{8}$$

in which

$$K^m_{ij} \equiv C^m_{ikjl}n_k n_l. \tag{9}$$

Under the assumption of thin coating and hence the constant variation of stresses and strains through the thickness of the coating, the volume average of a stress or strain function, being a function of surface direction, i.e. $F(\mathbf{n})$, over the domain of coating, $\Omega_2-\Omega_1$, can be calculated in a simple way. The details are provided in Appendix A.

Now since the total volumetric average of the stress field of the whole composite must be equal to σ_0, it follows that

$$\sigma_0 = f_0\sigma_m + f_1\sigma^f + f_2\langle\sigma^c\rangle_2 \tag{10}$$

in which f_0 is the volume fraction of the matrix. Substituting (3), (4) and (5) into (10) gives

$$\bar{\varepsilon} + f_1(\varepsilon^{11}+\varepsilon^{12}-\varepsilon^{*1}) + f_2\langle\varepsilon^{21}+\varepsilon^{22}-\varepsilon^{*2}\rangle_2 = \mathbf{0}. \tag{11}$$

In addition, the total volumetric average of the strain field of the whole composite, ε^T, can be expressed as

$$\begin{aligned}\varepsilon^T &= f_0(\varepsilon_0+\bar{\varepsilon}) + f_1(\varepsilon_0+\bar{\varepsilon}+\varepsilon^{11}+\varepsilon^{12}) + f_2\langle\varepsilon_0+\bar{\varepsilon}+\varepsilon^{21}+\varepsilon^{22}\rangle_2\\ &= \varepsilon_0+\bar{\varepsilon}+f_1(\varepsilon^{11}+\varepsilon^{12}) + f_2\langle\varepsilon^{21}+\varepsilon^{22}\rangle_2\\ &= \varepsilon_0 + f_1\varepsilon^{*1} + f_2\langle\varepsilon^{*2}\rangle_2\end{aligned} \tag{12}$$

in which the last equality is obtained through the use of (11).

Let us focus on the interface between a short fiber and a thin coating. Then the continuity of traction and displacement vectors requires at this interface that (Hill, 1961; Walpole, 1967)

$$\mathbf{u}^c - \mathbf{u}^f = \mathbf{0} \qquad \text{and} \qquad (\sigma^c-\sigma^f)\cdot\mathbf{n} = \mathbf{0}. \tag{13, 14}$$

Again $\mathbf{n}$ is a unit vector outward normal to the fiber surface. The displacement gradient tensor $u_{i,j}$ is discontinuous across the interface and the jump of $u_{i,j}$ across the interface can be expressed as

$$u^c_{i,j} - u^f_{i,j} = \lambda_i n_j$$

which in turn can be rewritten as

$$\varepsilon^c - \varepsilon^f = \tfrac{1}{2}(\boldsymbol{\lambda}\mathbf{n}+\mathbf{n}\boldsymbol{\lambda}). \tag{15}$$

Substituting (4), (5) and (15) into (14) yields

$$\lambda_p = K_{pi}^{c\ -1}(C_{ijkl}^{m} - C_{ijkl}^{c})\varepsilon_{kl}^{f} n_j - K_{pi}^{c\ -1} C_{ijkl}^{m}\varepsilon_{kl}^{*1} n_j \tag{16}$$

in which

$$K_{ij}^{c} \equiv C_{ikjl}^{c} n_k n_l. \tag{16a}$$

Substituting (15) and (16) into (5) gives

$$\sigma_{ij}^{c} = C_{ijkl}^{c}[\varepsilon_{kl}^{f} + K_{kp}^{c\ -1}(C_{pqrs}^{m} - C_{pqrs}^{c})\varepsilon_{rs}^{f} n_q n_l - K_{kp}^{c\ -1} C_{pqrs}^{m}\varepsilon_{rs}^{*1} n_q n_l]. \tag{17}$$

Let

$$P_{ijkl}^{c} = \tfrac{1}{4}(K_{ik}^{c\ -1} n_j n_l + K_{il}^{c\ -1} n_j n_k + K_{jk}^{c\ -1} n_i n_l + K_{jl}^{c\ -1} n_i n_k). \tag{17a}$$

Thus

$$\boldsymbol{\sigma}^{c} = \mathbf{D}^{1}\cdot\boldsymbol{\varepsilon}^{f} - \mathbf{D}^{2}\cdot\boldsymbol{\varepsilon}^{*1} \tag{18}$$

in which

$$\mathbf{D}^{1} = \mathbf{C}^{c} + \mathbf{C}^{c}\cdot\mathbf{P}^{c}\cdot(\mathbf{C}^{m} - \mathbf{C}^{c}) \quad \mathbf{D}^{2} = \mathbf{C}^{c}\cdot\mathbf{P}^{c}\cdot\mathbf{C}^{m}. \tag{18a}$$

From (5) and (18), we can get

$$\begin{aligned}\boldsymbol{\varepsilon}^{*2} &= (\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\boldsymbol{\sigma}^{c}\\ &= (\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{1}\cdot\boldsymbol{\varepsilon}^{f} - (\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{2}\cdot\boldsymbol{\varepsilon}^{*1}\\ &= (\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{1}\cdot(\boldsymbol{\varepsilon}_0 + \bar{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{11} + \boldsymbol{\varepsilon}^{12}) - (\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{2}\cdot\boldsymbol{\varepsilon}^{*1}.\end{aligned} \tag{19}$$

Substituting the above equation into (7) leads to

$$\begin{aligned}\boldsymbol{\varepsilon}^{12} &= f_{12}\langle\mathbf{S}^{0}\cdot\boldsymbol{\varepsilon}^{*2}\rangle_2\\ &= f_{12}\langle\mathbf{S}^{0}\cdot(\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{1}\rangle_2\cdot(\boldsymbol{\varepsilon}_0 + \bar{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{11} + \boldsymbol{\varepsilon}^{12})\\ &\quad - f_{12}\langle\mathbf{S}^{0}\cdot(\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{2}\rangle_2\cdot\boldsymbol{\varepsilon}^{*1}.\end{aligned} \tag{20}$$

The above equation can be rearranged to yield

$$\boldsymbol{\varepsilon}^{12} = \mathbf{R}_1\cdot(\boldsymbol{\varepsilon}_0 + \bar{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}^{11}) - \mathbf{R}_2\cdot\boldsymbol{\varepsilon}^{*1} \tag{21}$$

in which

$$\begin{gathered}\mathbf{R}_1 = (\mathbf{I} - \mathbf{D}^{3})^{-1}\cdot\mathbf{D}^{3} \quad \mathbf{R}_2 = (\mathbf{I} - \mathbf{D}^{3})^{-1}\cdot\mathbf{D}^{4}\\ \mathbf{D}^{3} = f_{12}\langle\mathbf{S}^{0}\cdot(\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{1}\rangle_2 \quad \mathbf{D}^{4} = f_{12}\langle\mathbf{S}^{0}\cdot(\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{2}\rangle_2\end{gathered} \tag{21a}$$

and $\mathbf{I}$ is the fourth rank identity tensor defined as

$$I_{pqrs} = \tfrac{1}{2}(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}). \tag{21b}$$

It is noted that in Appendix B a sample calculation of the averaged quantities such as $\langle\mathbf{S}^{0}\cdot(\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{1}\rangle_2$ is provided in detail. Similar quantities such as $\langle\mathbf{S}^{0}\cdot(\mathbf{C}^{c\ -1} - \mathbf{C}^{m\ -1})\cdot\mathbf{D}^{2}\rangle$, and others [like $\mathbf{R}_3$ and $\mathbf{R}_4$ in (23a)] below, can be calculated in the same way. From (5) we know

$$\varepsilon^{21}+\varepsilon^{22}-\varepsilon^{*2} = \mathbf{C}^{m-1}\cdot\sigma^{c}-\varepsilon^{0}-\bar{\varepsilon}. \qquad (22)$$

Substituting (18), (21) and (22) into (11) yields

$$\mathbf{F}_1\cdot\bar{\varepsilon}+\mathbf{F}_2\cdot\varepsilon^{11}+\mathbf{F}_3\cdot\varepsilon_0+\mathbf{F}_4\cdot\varepsilon^{*1} = \mathbf{0} \qquad (23)$$

in which

$$\mathbf{F}_1 = (1-f_2)\mathbf{I}+f_1\mathbf{R}_1+f_2\mathbf{R}_3+f_2\mathbf{R}_3\cdot\mathbf{R}_1 \quad \mathbf{F}_2 = f_1(\mathbf{I}+\mathbf{R}_1)+f_2\mathbf{R}_3+f_2\mathbf{R}_3\cdot\mathbf{R}_1$$
$$\mathbf{F}_3 = f_1\mathbf{R}_1+f_2\mathbf{R}_3+f_2\mathbf{R}_3\cdot\mathbf{R}_1-f_2\mathbf{I} \quad \mathbf{F}_4 = -f_1(\mathbf{I}+\mathbf{R}_2)-f_2\mathbf{R}_4-f_2\mathbf{R}_3\cdot\mathbf{R}_2$$
$$\mathbf{R}_3 = \langle\mathbf{C}^{m-1}\cdot\mathbf{D}^1\rangle_2 \quad \mathbf{R}_4 = \langle\mathbf{C}^{m-1}\cdot\mathbf{D}^2\rangle_2. \qquad (23a)$$

Thus, eqn (23) can be rearranged to yield

$$\bar{\varepsilon} = -\mathbf{F}_1^{-1}\cdot[\mathbf{F}_2\cdot\varepsilon^{11}+\mathbf{F}_3\cdot\varepsilon_0+\mathbf{F}_4\cdot\varepsilon^{*1}]. \qquad (24)$$

Substituting (24) into (21) yields

$$\begin{aligned}\varepsilon^{12} &= \mathbf{R}_1\cdot[\varepsilon_0+\varepsilon^{11}-\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\varepsilon^{11}+\mathbf{F}_3\cdot\varepsilon_0+\mathbf{F}_4\cdot\varepsilon^{*1})]-\mathbf{R}_2\cdot\varepsilon^{*1}\\ &= \mathbf{H}_1\cdot\varepsilon_0+\mathbf{H}_2\cdot\varepsilon^{11}+\mathbf{H}_3\cdot\varepsilon^{*1}\end{aligned} \qquad (25)$$

in which

$$\mathbf{H}_1 = \mathbf{R}_1-\mathbf{R}_1\cdot\mathbf{F}_1^{-1}\cdot\mathbf{F}_3 \quad \mathbf{H}_2 = \mathbf{R}_1-\mathbf{R}_1\cdot\mathbf{F}_1^{-1}\cdot\mathbf{F}_2 \quad \mathbf{H}_3 = -\mathbf{R}_2-\mathbf{R}_1\cdot\mathbf{F}_1^{-1}\cdot\mathbf{F}_4. \qquad (25a)$$

From (4), we find

$$\varepsilon^{*1} = \mathbf{A}\cdot\mathbf{T}\cdot(\varepsilon_0+\bar{\varepsilon}+\varepsilon^{12}) \qquad (26)$$

in which

$$\mathbf{A} = (\mathbf{I}-\mathbf{C}^{m-1}\cdot\mathbf{C}^{f}) \quad \mathbf{T} = (\mathbf{I}+\mathbf{S}\cdot\mathbf{C}^{m-1}\cdot\mathbf{C}^{f}-\mathbf{S})^{-1}. \qquad (26a)$$

Substituting (6) into (24) and (25) yields

$$\begin{aligned}\bar{\varepsilon} &= -\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}\cdot\varepsilon^{*1}+\mathbf{F}_3\cdot\varepsilon_0+\mathbf{F}_4\cdot\varepsilon^{*1})\\ &= -\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\varepsilon^{*1}-\mathbf{F}_1^{-1}\cdot\mathbf{F}_3\cdot\varepsilon_0\end{aligned} \qquad (27)$$

and

$$\varepsilon^{12} = \mathbf{H}_1\cdot\varepsilon_0+(\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\varepsilon^{*1}. \qquad (28)$$

Substituting (27) and (28) into (26) yields

$$\begin{aligned}\varepsilon^{*1} &= \mathbf{A}\cdot\mathbf{T}\cdot[\varepsilon_0-\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\varepsilon^{*1}-\mathbf{F}_1^{-1}\cdot\mathbf{F}_3\cdot\varepsilon_0+\mathbf{H}_1\cdot\varepsilon_0+(\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\varepsilon^{*1}]\\ &= \mathbf{E}_1\cdot\varepsilon_0+\mathbf{E}_2\cdot\varepsilon^{*1}\end{aligned} \qquad (29)$$

in which

$$\mathbf{E}_1 = \mathbf{A}\cdot\mathbf{T}\cdot[\mathbf{I}-\mathbf{F}_1^{-1}\cdot\mathbf{F}_3+\mathbf{H}_1] \quad \mathbf{E}_2 = \mathbf{A}\cdot\mathbf{T}\cdot[\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3-\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)]. \qquad (29a)$$

Rearranging (29) gives

$$\boldsymbol{\varepsilon}^{*1} = (\mathbf{I}-\mathbf{E}_2)^{-1}\cdot\mathbf{E}_1\cdot\boldsymbol{\varepsilon}_0 = \mathbf{E}_3\cdot\boldsymbol{\varepsilon}_0 \tag{30}$$

in which

$$\mathbf{E}_3 = (\mathbf{I}-\mathbf{E}_2)^{-1}\cdot\mathbf{E}_1. \tag{30a}$$

Substituting (30) into (27) and (28) yields

$$\begin{aligned}\bar{\boldsymbol{\varepsilon}} &= -\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\mathbf{E}_3\cdot\boldsymbol{\varepsilon}_0-\mathbf{F}_1^{-1}\cdot\mathbf{F}_3\cdot\boldsymbol{\varepsilon}_0\\ &= \mathbf{E}_4\cdot\boldsymbol{\varepsilon}_0\end{aligned} \tag{31}$$

and

$$\begin{aligned}\boldsymbol{\varepsilon}_{12} &= \mathbf{H}_1\cdot\boldsymbol{\varepsilon}_0+(\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\mathbf{E}_3\cdot\boldsymbol{\varepsilon}_0\\ &= \mathbf{E}_5\cdot\boldsymbol{\varepsilon}_0\end{aligned} \tag{32}$$

in which

$$\mathbf{E}_4 = -\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\mathbf{E}_3-\mathbf{F}_1^{-1}\cdot\mathbf{F}_3 \quad \mathbf{E}_5 = \mathbf{H}_1+(\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\mathbf{E}_3. \tag{32a}$$

Substituting (6), (19), (30), (31) and (32) into (12) yields

$$\boldsymbol{\varepsilon}^{\mathrm{T}} = [\mathbf{I}+f_1\mathbf{E}_3+f_2\mathbf{R}_5\cdot(\mathbf{I}+\mathbf{E}_4+\mathbf{E}_5+\mathbf{S}\cdot\mathbf{E}_3)-f_2\mathbf{R}_6\cdot\mathbf{E}_3]\cdot\boldsymbol{\varepsilon}_0 \tag{33}$$

in which

$$\mathbf{R}^5 = \langle(\mathbf{C}^{c\ 1}-\mathbf{C}^{m\ 1})\cdot\mathbf{D}^1\rangle_2 \quad \mathbf{R}^6 = \langle(\mathbf{C}^{c\ 1}-\mathbf{C}^{m\ 1})\cdot\mathbf{D}^2\rangle_2. \tag{33a}$$

Now define the effective elastic moduli $\mathbf{C}^*$ of the composite according to

$$\boldsymbol{\sigma}_0 = \mathbf{C}^*\cdot\boldsymbol{\varepsilon}^{\mathrm{T}}. \tag{34}$$

Then, combining (1) and (33) with (34), the effective elastic moduli $\mathbf{C}^*$ are given as

$$\mathbf{C}^* = \mathbf{C}^{m}\cdot[\mathbf{I}+f_1\mathbf{E}_3+f_2\mathbf{R}_5\cdot(\mathbf{I}+\mathbf{E}_4+\mathbf{E}_5+\mathbf{S}\cdot\mathbf{E}_3)-f_2\mathbf{R}_6\cdot\mathbf{E}_3]^{-1}. \tag{35}$$

EFFECTIVE COEFFICIENTS OF THERMAL EXPANSION

As a parallel to the above derivation, to evaluate the effective CTE (coefficients of thermal expansion) of the composites with coated fibers, let us first subject the matrix alone to a uniform temperature field ΔT. The uniform thermal strains $\boldsymbol{\varepsilon}_0$ produced due to such a temperature field are

$$\boldsymbol{\varepsilon}_0 = \Delta T\boldsymbol{\alpha}_{\mathrm{m}} \tag{36}$$

in which $\boldsymbol{\alpha}_{\mathrm{m}}$ is the second order tensor of the coefficients of thermal expansion of the matrix. It is noted that due to free expansion and no applied external loads, there are no stresses in the matrix, i.e. $\boldsymbol{\sigma}_0 = \mathbf{0}$. When there are ellipsoidal coated fibers present in the matrix, a perturbed stress field $\tilde{\boldsymbol{\sigma}}(\mathbf{x})$ is induced and the volumetric average of the perturbed stress field of the matrix, $\bar{\boldsymbol{\sigma}}$, is again given by eqn (2). From Eshelby's equivalence inclusion principle, one now has, in the domain of a typical fiber, Ω_1:

$$\boldsymbol{\sigma}^{\mathrm{f}} = \mathbf{C}^{\mathrm{f}}\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{11}+\boldsymbol{\varepsilon}^{12}-\boldsymbol{\varepsilon}^{\mathrm{T1}})$$
$$= \mathbf{C}^{\mathrm{m}}\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{11}+\boldsymbol{\varepsilon}^{12}-\boldsymbol{\varepsilon}^{**1}) \tag{37}$$

and in the domain of the coating of a typical fiber, $\Omega_2-\Omega_1$:

$$\boldsymbol{\sigma}^{\mathrm{c}} = \mathbf{C}^{\mathrm{c}}\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{22}+\boldsymbol{\varepsilon}^{21}-\boldsymbol{\varepsilon}^{\mathrm{T2}})$$
$$= \mathbf{C}^{\mathrm{m}}\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{22}+\boldsymbol{\varepsilon}^{21}-\boldsymbol{\varepsilon}^{**2}) \tag{38}$$

in which

$$\boldsymbol{\varepsilon}^{\mathrm{T1}} = (\boldsymbol{\alpha}_{\mathrm{f}}-\boldsymbol{\alpha}_{\mathrm{m}})\Delta T \quad \boldsymbol{\varepsilon}^{\mathrm{T2}} = (\boldsymbol{\alpha}_{\mathrm{c}}-\boldsymbol{\alpha}_{\mathrm{m}})\Delta T \tag{39}$$

and $\boldsymbol{\alpha}_{\mathrm{f}}$ and $\boldsymbol{\alpha}_{\mathrm{c}}$ are the tensors of the coefficients of thermal expansion of the fiber and the coating. It is noted that the relationships for $\boldsymbol{\varepsilon}^{11}$ and $\boldsymbol{\varepsilon}^{12}$, eqns (6)–(9), are again valid. Further, the traction continuity and jump condition of displacement gradient given by eqns (14) and (15), hold here. For easy reference, they are repeated as follows:

$$(\boldsymbol{\sigma}^{\mathrm{c}}-\boldsymbol{\sigma}^{\mathrm{f}})\cdot\mathbf{n} = \mathbf{0} \tag{14}$$

$$\boldsymbol{\varepsilon}^{\mathrm{c}}-\boldsymbol{\varepsilon}^{\mathrm{f}} = (\boldsymbol{\varepsilon}_0+\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{22}+\boldsymbol{\varepsilon}^{21})-(\boldsymbol{\varepsilon}_0+\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{11}+\boldsymbol{\varepsilon}^{12}) = \tfrac{1}{2}(\boldsymbol{\lambda}\mathbf{n}+\mathbf{n}\boldsymbol{\lambda}) \tag{15}$$

in which $\boldsymbol{\varepsilon}^{\mathrm{c}}$ and $\boldsymbol{\varepsilon}^{\mathrm{f}}$ are total strains of coating and fiber. Now substituting (37), (38) and (15) into (14) yields

$$\lambda_p = K^{\mathrm{c}\,-1}_{pi}[(C^{\mathrm{m}}_{ijkl}-C^{\mathrm{c}}_{ijkl})(\bar{\varepsilon}_{kl}+\varepsilon^{11}_{kl}+\varepsilon^{12}_{kl})n_j-C^{\mathrm{m}}_{ijkl}\varepsilon^{**1}_{kl}n_j+C^{\mathrm{c}}_{ijkl}\varepsilon^{\mathrm{T2}}_{kl}n_j] \tag{40}$$

in which K^{c}_{pi} is defined as eqn (16a). Substituting (15) and (40) into (38) gives

$$\boldsymbol{\sigma}^{\mathrm{c}} = \mathbf{D}^1\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{11}+\boldsymbol{\varepsilon}^{12})-\mathbf{D}^2\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{D}^5\cdot\boldsymbol{\varepsilon}^{\mathrm{T2}} \tag{41}$$

in which $\mathbf{D}^1$ and $\mathbf{D}^2$ are given by eqn (18a) and

$$\mathbf{D}^5 = \mathbf{C}^{\mathrm{c}}\cdot\mathbf{P}^{\mathrm{c}}\cdot\mathbf{C}^{\mathrm{c}}-\mathbf{C}^{\mathrm{c}} \tag{41a}$$

with $\mathbf{P}^{\mathrm{c}}$ defined by (17a).

The total stress average over the whole domain is again given by eqn (11), and the total strain average is

$$\boldsymbol{\varepsilon}^{\mathrm{T}} = \boldsymbol{\alpha}_{\mathrm{m}}\Delta T+f_1\boldsymbol{\varepsilon}^{**1}+f_2\langle\boldsymbol{\varepsilon}^{**2}\rangle_2. \tag{42}$$

From (38) and (41), we can get

$$\boldsymbol{\varepsilon}^{**2} = (\mathbf{C}^{\mathrm{c}\,-1}-\mathbf{C}^{\mathrm{m}\,-1})\cdot\boldsymbol{\sigma}^{\mathrm{c}}+\boldsymbol{\varepsilon}^{\mathrm{T2}}$$
$$= (\mathbf{C}^{\mathrm{c}\,-1}-\mathbf{C}^{\mathrm{m}\,-1})\cdot\mathbf{D}^1\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{11}+\boldsymbol{\varepsilon}^{12})-(\mathbf{C}^{\mathrm{c}\,-1}-\mathbf{C}^{\mathrm{m}\,-1})\cdot\mathbf{D}^2\cdot\boldsymbol{\varepsilon}^{**1}$$
$$+[(\mathbf{C}^{\mathrm{c}\,-1}-\mathbf{C}^{\mathrm{m}\,-1})\cdot\mathbf{D}^5+\mathbf{I}]\cdot\boldsymbol{\varepsilon}^{\mathrm{T2}}. \tag{43}$$

Thus eqn (7) can be rewritten as

$$\boldsymbol{\varepsilon}^{12} = \mathbf{R}_1\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{11})-\mathbf{R}_2\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{R}_7\cdot\boldsymbol{\varepsilon}^{\mathrm{T2}} \tag{44}$$

in which $\mathbf{R}_1$ and $\mathbf{R}_2$ are given by (21a), and

$$\mathbf{R}_7 = (\mathbf{I}-\mathbf{D}^3)^{-1}\cdot\mathbf{D}^6 \quad \mathbf{D}^6 = f_{12}\langle \mathbf{S}^0\cdot[(\mathbf{C}^{c^{-1}}-\mathbf{C}^{m^{-1}})\cdot\mathbf{D}^5+\mathbf{I}]\rangle_2. \tag{44a}$$

From (38) we know that

$$\boldsymbol{\varepsilon}^{21}+\boldsymbol{\varepsilon}^{22}-\boldsymbol{\varepsilon}^{**2} = \mathbf{C}^{m^{-1}}\cdot\boldsymbol{\sigma}^c-\bar{\boldsymbol{\varepsilon}}. \tag{45}$$

Substituting (15), (44) and (45) into (11) yields

$$\mathbf{F}_1\cdot\bar{\boldsymbol{\varepsilon}}+\mathbf{F}_2\cdot\boldsymbol{\varepsilon}^{11}+\mathbf{F}_4\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{F}_5\cdot\boldsymbol{\varepsilon}^{T2} = \mathbf{0} \tag{46}$$

in which $\mathbf{F}_1$ and $\mathbf{F}_2$ are given by (23a), and

$$\mathbf{F}_5 = f_1\mathbf{R}_7+f_2\mathbf{R}_8+f_2\mathbf{R}_3\cdot\mathbf{R}_7 \quad \mathbf{R}_8 = \langle\mathbf{C}^{m^{-1}}\cdot\mathbf{D}^5\rangle_2. \tag{46a}$$

Equation (46) can be rearranged to give

$$\bar{\boldsymbol{\varepsilon}} = -\mathbf{F}_1^{-1}\cdot[\mathbf{F}_2\cdot\boldsymbol{\varepsilon}^{11}+\mathbf{F}_4\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{F}_5\cdot\boldsymbol{\varepsilon}^{T2}]. \tag{47}$$

Substituting (47) into (44) yields

$$\begin{aligned}\boldsymbol{\varepsilon}^{12} &= \mathbf{R}_1\cdot[\boldsymbol{\varepsilon}^{11}-\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\boldsymbol{\varepsilon}^{11}+\mathbf{F}_4\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{F}_5\cdot\boldsymbol{\varepsilon}^{T2})]-\mathbf{R}_2\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{R}_7\cdot\boldsymbol{\varepsilon}^{T2}\\ &= \mathbf{H}_2\cdot\boldsymbol{\varepsilon}^{11}+\mathbf{H}_3\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{H}_4\cdot\boldsymbol{\varepsilon}^{T2}\end{aligned} \tag{48}$$

in which $\mathbf{H}_2$ and $\mathbf{H}_3$ are given by (25a), and

$$\mathbf{H}_4 = \mathbf{R}_7-\mathbf{R}_1\cdot\mathbf{F}_1^{-1}\cdot\mathbf{F}_5. \tag{48a}$$

From (37), we have

$$\boldsymbol{\varepsilon}^{**1} = \mathbf{A}\cdot\mathbf{T}\cdot(\bar{\boldsymbol{\varepsilon}}+\boldsymbol{\varepsilon}^{12})+(\mathbf{A}\cdot\mathbf{T}\cdot\mathbf{S}+\mathbf{I})\cdot\mathbf{C}^{m^{-1}}\cdot\mathbf{C}^f\cdot\boldsymbol{\varepsilon}^{T1} \tag{49}$$

in which $\mathbf{A}$ and $\mathbf{T}$ are given by (26a). Substituting (6) into (47) and (48) yields

$$\bar{\boldsymbol{\varepsilon}} = -\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\boldsymbol{\varepsilon}^{**1}-\mathbf{F}_1^{-1}\cdot\mathbf{F}_5\cdot\boldsymbol{\varepsilon}^{T2} \tag{50}$$

and

$$\boldsymbol{\varepsilon}^{12} = \mathbf{H}_4\cdot\boldsymbol{\varepsilon}^{T2}+(\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\boldsymbol{\varepsilon}^{**1}. \tag{51}$$

Substituting (50) and (51) into (49) yields

$$\begin{aligned}\boldsymbol{\varepsilon}^{**1} &= \mathbf{A}\cdot\mathbf{T}\cdot[-\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\boldsymbol{\varepsilon}^{**1}-\mathbf{F}_1\cdot\mathbf{F}_5\cdot\boldsymbol{\varepsilon}^{T2}+\mathbf{H}_4\cdot\boldsymbol{\varepsilon}^{T2}\\ &\quad+(\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\boldsymbol{\varepsilon}^{**1}]+(\mathbf{A}\cdot\mathbf{T}\cdot\mathbf{S}+\mathbf{I})\cdot\mathbf{C}^{m^{-1}}\cdot\mathbf{C}^f\cdot\boldsymbol{\varepsilon}^{T1}\\ &= \mathbf{E}_2\cdot\boldsymbol{\varepsilon}^{**1}+\mathbf{E}_6\cdot\boldsymbol{\varepsilon}^{T2}+\mathbf{E}_7\cdot\boldsymbol{\varepsilon}^{T1}\end{aligned} \tag{52}$$

in which $\mathbf{E}_2$ is given by (29a), and

$$\mathbf{E}_6 = \mathbf{A}\cdot\mathbf{T}\cdot[\mathbf{H}_4-\mathbf{F}_1^{-1}\cdot\mathbf{F}_5] \quad \mathbf{E}_7 = (\mathbf{A}\cdot\mathbf{T}\cdot\mathbf{S}+\mathbf{I})\cdot\mathbf{C}^{m^{-1}}\cdot\mathbf{C}^f. \tag{52a}$$

Rearrange (52) to give

$$\varepsilon^{**1} = (\mathbf{I}-\mathbf{E}_2)^{-1}\cdot\mathbf{E}_6\cdot\varepsilon^{T2}+(\mathbf{I}-\mathbf{E}_2)^{-1}\cdot\mathbf{E}_7\cdot\varepsilon^{T1}$$
$$= \mathbf{E}_8\cdot\varepsilon^{T2}+\mathbf{E}_9\cdot\varepsilon^{T1} \tag{53}$$

in which

$$\mathbf{E}_8 = (\mathbf{I}-\mathbf{E}_2)^{-1}\cdot\mathbf{E}_6 \quad \mathbf{E}_9 = (\mathbf{I}-\mathbf{E}_2)^{-1}\cdot\mathbf{E}_7. \tag{53a}$$

Substituting (53) into (50) and (51) yields

$$\bar{\varepsilon} = \mathbf{E}_{10}\cdot\varepsilon^{T2}+\mathbf{E}_{11}\cdot\varepsilon^{T1} \quad \text{and} \quad \varepsilon_{12} = \mathbf{E}_{12}\cdot\varepsilon^{T2}+\mathbf{E}_{13}\cdot\varepsilon^{T1} \tag{54}$$

in which

$$\mathbf{E}_{10} = -\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\mathbf{E}_8-\mathbf{F}_1^{-1}\cdot\mathbf{F}_5 \quad \mathbf{E}_{11} = -\mathbf{F}_1^{-1}\cdot(\mathbf{F}_2\cdot\mathbf{S}+\mathbf{F}_4)\cdot\mathbf{E}_9$$
$$\mathbf{E}_{12} = \mathbf{H}_4+(\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\mathbf{E}_8 \quad \mathbf{E}_{13} = (\mathbf{H}_2\cdot\mathbf{S}+\mathbf{H}_3)\cdot\mathbf{E}_9. \tag{55}$$

Substituting (6), (43), (53), (54) and (55) into (42) yields

$$\varepsilon^{T} = \alpha_m\Delta T+\mathbf{E}_{14}\cdot\varepsilon^{T2}+\mathbf{E}_{15}\cdot\varepsilon^{T1} \tag{56}$$

in which $\mathbf{R}_5$ and $\mathbf{R}_6$ are given by (33a), and

$$\mathbf{E}_{14} = f_1\mathbf{E}_8+f_2[\mathbf{R}_5\cdot(\mathbf{E}_{10}+\mathbf{E}_{12}+\mathbf{S}\cdot\mathbf{E}_8)-\mathbf{R}_6\cdot\mathbf{E}_8+\mathbf{R}_9]$$
$$\mathbf{E}_{15} = f_1\mathbf{E}_9+f_2[\mathbf{R}_5\cdot(\mathbf{E}_{11}+\mathbf{E}_{13}+\mathbf{S}\cdot\mathbf{E}_9)-\mathbf{R}_6\cdot\mathbf{E}_9]$$
$$\mathbf{R}^9 = \langle(\mathbf{C}^{c\,-1}-\mathbf{C}^{m\,-1})\cdot\mathbf{D}^2+\mathbf{I}\rangle_2. \tag{56a}$$

Thus the effective coefficient tensor of thermal expansion α^* is given as

$$\alpha^* = \frac{\varepsilon^T}{\Delta T} = \alpha_m+\mathbf{E}_{14}\cdot(\alpha_c-\alpha_m)+\mathbf{E}_{15}\cdot(\alpha_f-\alpha_m). \tag{57}$$

NUMERICAL EXAMPLES AND DISCUSSION

A comparison study with existing works and a parametric study were conducted based on the formulation developed in the previous sections.

Example 1

The first example is a comparison of elastic moduli for a composite with long continuous coated fibers based on the present work with those of Pagano and Tandon's composite cylinder assemblage model (1988). The materials (matrix, fiber and coating) are all isotropic and their properties are listed as follows:

fiber: Nicalon

E(GPa) = 200, G(GPa) = 77, volume fraction = 0.6;

matrix: BMAS

E(GPa) = 106, G(GPa) = 43.

In addition, the Poisson ratio of coating material is 0.31, however, the coating Young's modulus and the ratio of coating thickness to radius of coated fiber were selected as

Table 1. Comparisons of the present model with Pagano and Tandon's work (1988) on the effective elastic moduli of a composite with long coated fibers

Coating thickness / Fiber radius	Coating modulus GPa	E_{11} GPa	E_{22} GPa	γ_{12} GPa	G_{23} GPa
0	—	162.528	151.964	60.614	59.814
		(162.530)	(152.840)	(60.615)	(60.357)
	0.345	160.830	40.391	18.495	15.118
		(160.833)	(45.148)	(18.361)	(18.010)
0.01291	3.45	160.920	106.460	43.616	40.848
		(160.922)	(109.741)	(43.611)	(42.811)
	34.5	161.411	147.090	58.696	57.684
		(161.412)	(148.031)	(58.697)	(58.263)
	0.345	153.982	21.620	10.932	8.114
		(153.986)	(22.891)	(10.149)	(9.232)
0.06455	3.45	154.292	53.835	23.410	20.114
		(154.296)	(58.332)	(23.057)	(22.850)
	34.5	156.798	130.090	52.019	50.361
		(156.799)	(131.901)	(52.043)	(51.464)
	0.345	144.994	15.377	8.032	5.766
		(145.002)	(14.624)	(6.778)	(5.826)
0.1291	3.45	145.557	34.792	15.665	12.871
		(145.560)	(37.123)	(14.852)	(14.500)
	34.5	150.704	113.016	45.280	43.162
		(150.704)	(115.995)	(45.307)	(44.914)

Numbers in ( ) are picked from Pagano and Tandon (1988).

parameters. The results are documented in Table 1. As follows, it is shown that even for a moderately thick coating, good consistency is still reached.

Example 2

The second example compares our prediction of thermoelastic properties of a composite reinforced by long continuous coated fibers with the work by Benveniste *et al.* (1989). The material systems 1, 3 and 4 are picked from Table 1 of the work by Benveniste *et al.* (1989). They are now redefined as systems 1, 2 and 3. The results are documented in Table

Table 2. Comparisons of the present model with Benveniste *et al.*'s work (1989) on the effective thermoelastic moduli of a composite with long coated fibers

	Materials	Benveniste *et al.* (1989)	Present
	1	1.255	1.255
E_A/E_m	2	1.236	1.236
	3	2.379	2.380
	1	1.188	1.189
μ_A/μ_m	2	1.171	1.171
	3	1.655	1.654
	1	3.224×10^{-6}	3.225×10^{-6}
$\alpha_T(^\circ C^{-1})$	2	10.09×10^{-6}	10.09×10^{-6}
	3	7.638×10^{-6}	7.598×10^{-6}
	1	3.332×10^{-6}	3.333×10^{-6}
$\alpha_A(^\circ C^{-1})$	2	9.071×10^{-6}	9.039×10^{-6}
	3	5.998×10^{-6}	5.979×10^{-6}

E_A: Effective longitudinal Young's modulus.
μ_A: Effective longitudinal shear modulus.
α_T: Effective transverse coefficient of thermal expansion.
α_A: Effective longitudinal coefficient of thermal expansion.

2. Excellent consistency is reached, as expected, since both were based on Mori-Tanaka's method and the coatings of these three systems were all thin (the ratio of coating thickness to fiber radius is 0.02056 for system 1 and 0.01329 for systems 2 and 3).

Example 3

In the above two examples, it is seen that the present work gives excellent agreement with previous analytical work for continuous thin coated fiber. In this example we compare our solution to recent work by Tong and Jasiuk (1990), where they considered the effect of coating thickness of the effective coefficients of thermal expansion for a composite reinforced with coated spherical particles. The result is shown in Fig. 1. It is seen that in this extreme case our solution was also valid even when the layer of coating was thick (say, the volume ratio of coating layer to matrix, l/m, is 0.2). In fact a further confirmation on the validity of the present solution for the case of thick coating fibers has been shown by the authors (1991) after a comparison with the work by Pagano and Tandon (1990) on multidirectional continuous fibers.

Example 4

The fourth example is a parametric study for a short coated fiber composite. The materials of matrix, fiber and coating are all isotropic. Denoting:

E_m, E_f, E_c = Young's moduli of matrix, fiber and coating.
ν_m, ν_f, ν_c = Poisson's ratio of matrix, fiber and coating.
α_m, α_f, α_c = coefficients of thermal expansion of matrix, fiber and coating.
f_0, f_1, f_2 = volume fraction of matrix, fiber and coating.
l, d = long and short axis of the ellipsoidal fiber.

The fixed material properties are

$$E_f/E_m = 5, \quad \nu_c = \nu_f = 0.3, \quad \nu_m = 0.4, \alpha_f/\alpha_m = 5, \quad f_1 = 0.5.$$

The results are presented in Figs 2–15. In Figs 2–8, the effective thermoelastic properties of the composite are plotted against the ratio of coating volume fraction to fiber volume

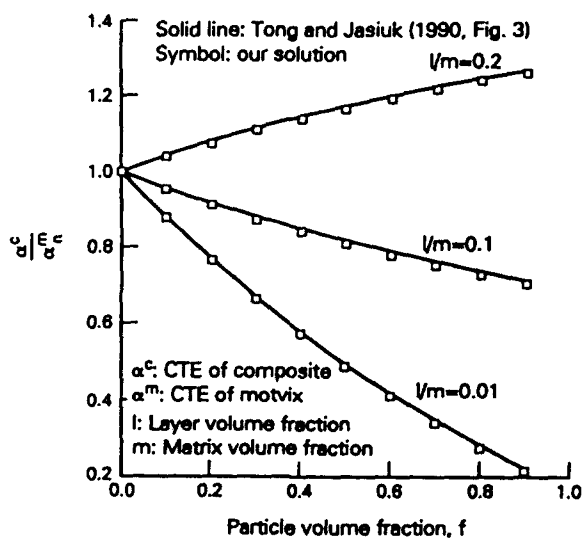

Fig. 1. Comparison of present model with Tong and Jasiuk's work (1990) on dimensionless parameter of the effective thermal coefficient, α_c/α_m, of a composite with spherical coated fibers.

fraction, f_2/f_1, with the aspect ratio of fiber, l/d, being chosen as 5. However, in Figs 9–15, the effective thermoelastic properties of the composite are plotted against the aspect ratio of the fiber, l/d, with $f_2/f_1 = 0.04$. In all these figures, the ratio of Young's coating modulus to Young's matrix modulus, E_c/E_m, is selected as the parameter. Besides, in Figs 6, 7, 13 and 14, the ratio of α_c/α_m is a combined parameter together with E_c/E_m.

As follows from Figs 2–6, all the dimensionless moduli (E_{11}/E_m, E_{22}/E_m, G_{12}/E_m and G_{23}/E_m) except ν_{12} increase monotonely as the parameter E_c/E_m increases. It is also noted that except ν_{12} all these moduli drop drastically near the origin as expected since the value of zero of E_c/E_m represents a debonding zone. Further, all these moduli including ν_{12} are less sensitive to large E_c/E_m than to small E_c/E_m. It also follows from Figs 7 and 8 that when the parameter α_c/α_m is small, α_{11}/α_m is less sensitive to the parameter E_c/E_m than is α_{22}/α_m.

As for the behavior of the thermoelastic moduli versus the fiber aspect ratio l/d, shown in Figs 9–15, this has been known in other works and is not discussed further here except

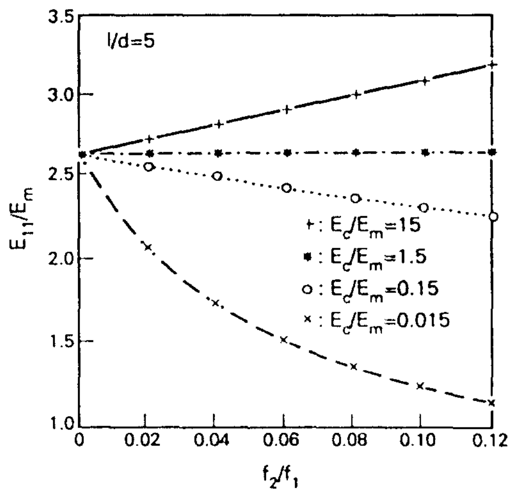

Fig. 2. Relationship of dimensionless longitudinal Young's modulus versus ratio of coating volume fraction to fiber volume fraction.

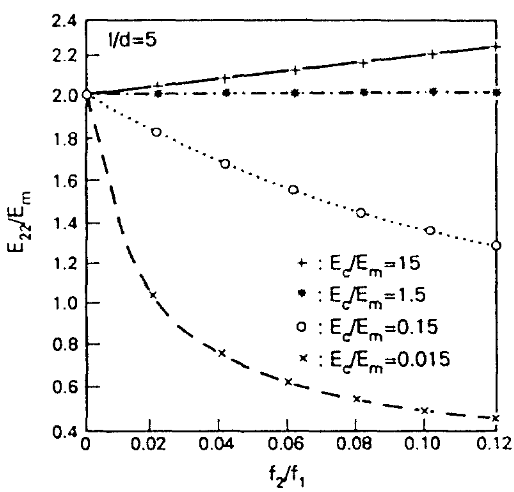

Fig. 3. Relationship of dimensionless transverse Young's modulus versus ratio of coating volume fraction to fiber volume fraction.

that it is worthwhile to notice the interesting behavior of Poisson's ratio ν_{12} near the origin (very short fiber), as shown in Fig. 13.

CONCLUDING REMARKS

In this work, the formulae for effective thermoelastic properties of a thin coated short fiber composite were derived under the assumption of thin coating and constant variation of stresses and strains through the thickness of the coating. In the regime of thin coating, the numerical comparisons with existing works showed excellent consistency. As for the thick coating, a comparison of our special case with currently available work showed another satisfaction. A parametric study was also conducted for the purpose of illustration. The present work should be extendable to many kinds of fibers and randomly oriented systems. They are currently under investigation and will appear soon.

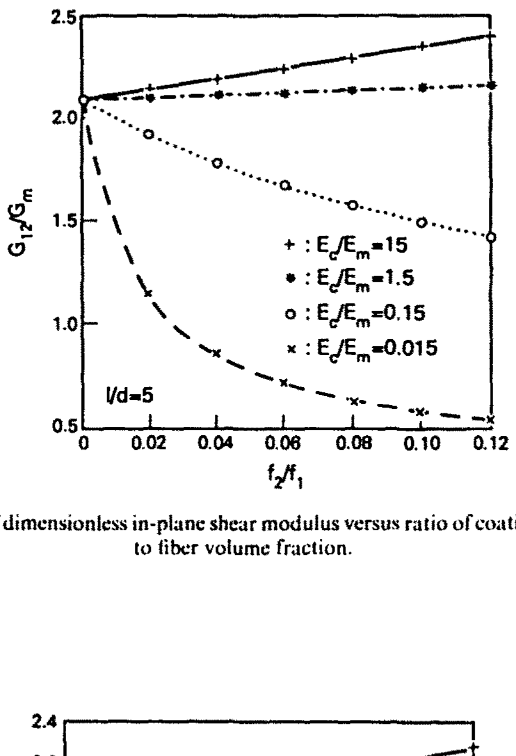

Fig. 4. Relationship of dimensionless in-plane shear modulus versus ratio of coating volume fraction to fiber volume fraction.

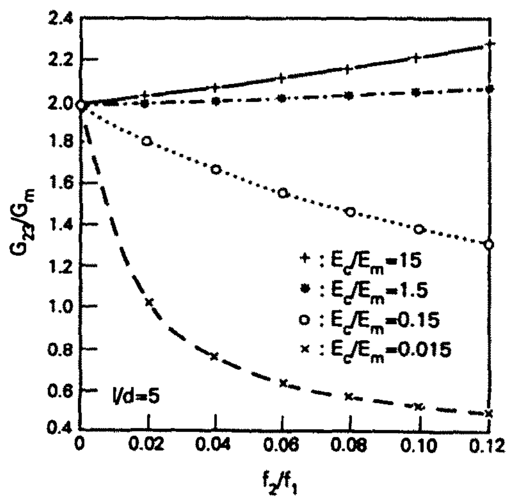

Fig. 5. Relationship of dimensionless transverse shear modulus versus ratio of coating volume fraction to fiber volume fraction.

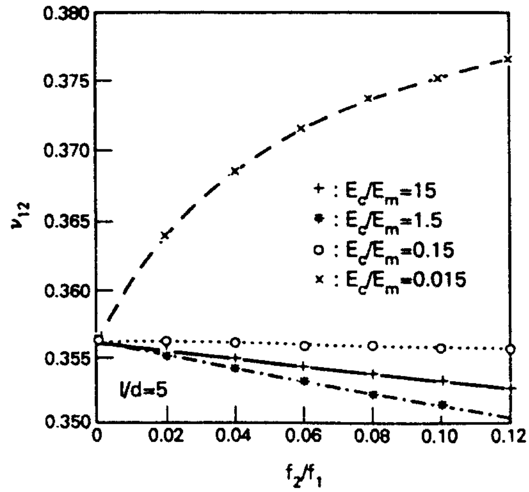

Fig. 6. Relationship of Poisson's ratio versus ratio of coating volume fraction to fiber volume fraction.

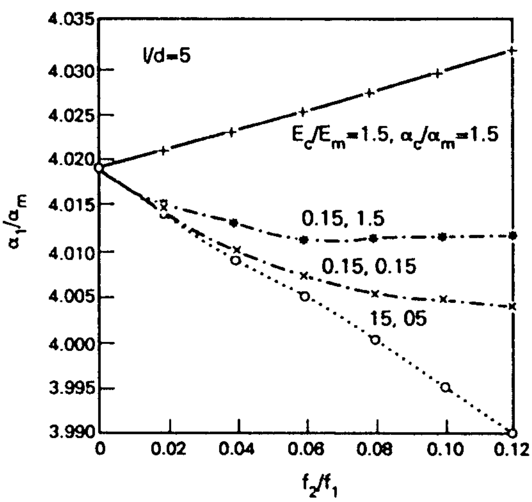

Fig. 7. Relationship of dimensionless longitudinal thermal expansion coefficient versus ratio of coating volume fraction to fiber volume fraction.

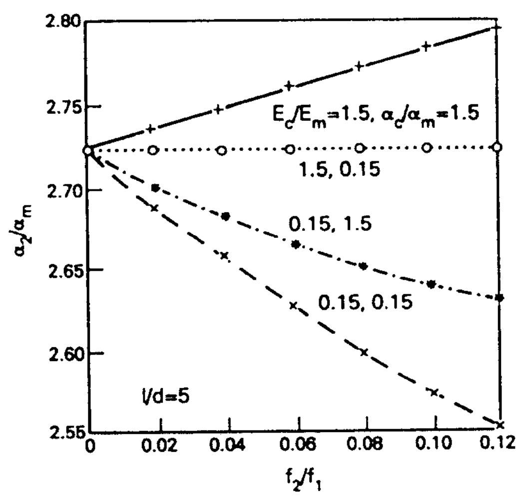

Fig. 8. Relationship of dimensionless transverse thermal expansion coefficient versus ratio of coating volume fraction to fiber volume fraction.

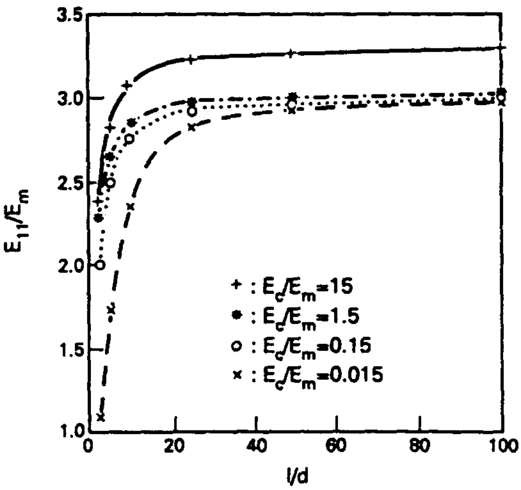

Fig. 9. Relationship of dimensionless longitudinal Young's modulus versus fiber aspect ratio.

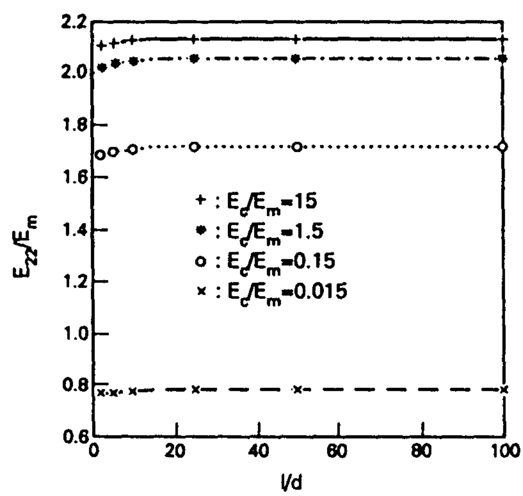

Fig. 10. Relationship of dimensionless transverse Young's modulus versus fiber aspect ratio.

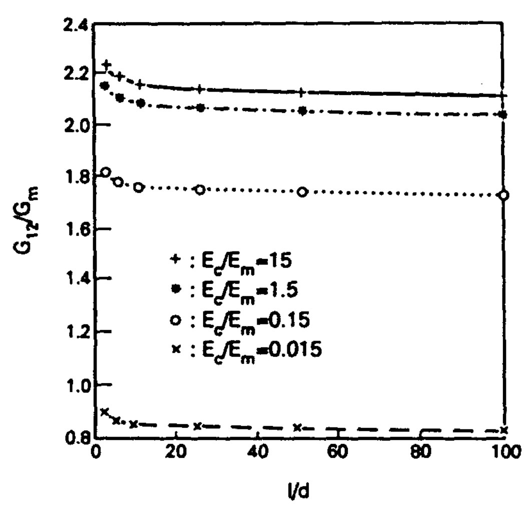

Fig. 11. Relationship of dimensionless in-plane shear modulus versus fiber aspect ratio.

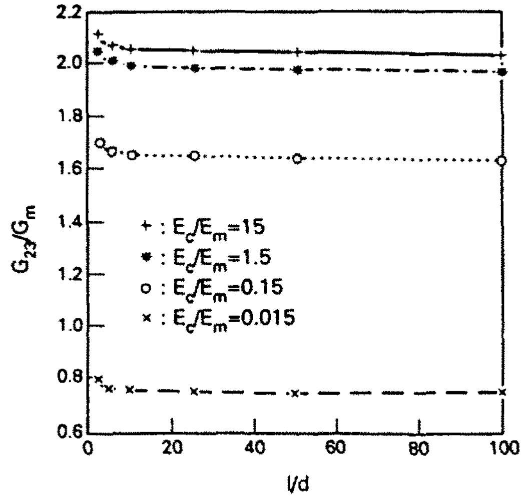

Fig. 12. Relationship of dimensionless transverse shear modulus versus fiber aspect ratio.

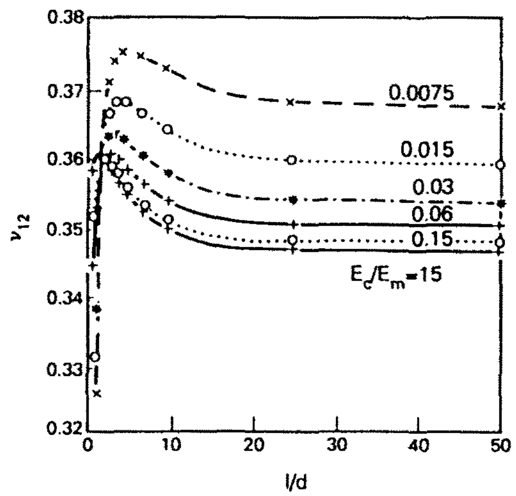

Fig. 13. Relationship of Poisson's ratio versus fiber aspect ratio.

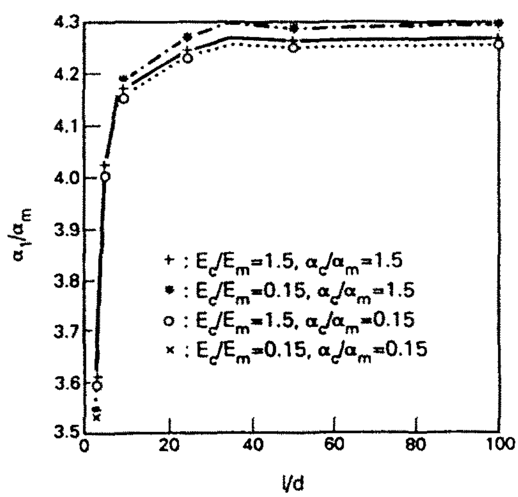

Fig. 14. Relationship of dimensionless longitudinal thermal expansion coefficient versus fiber aspect ratio.

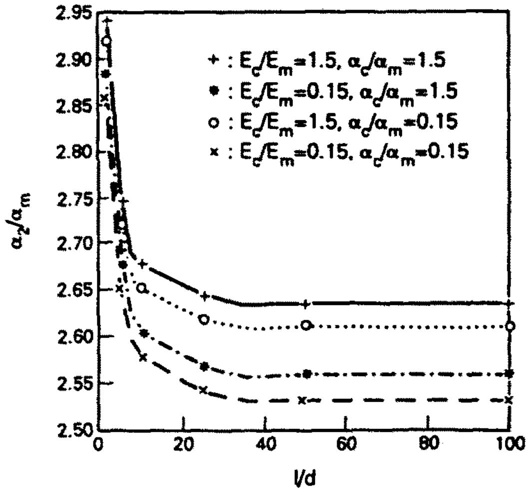

Fig. 15. Relationship of dimensionless transverse thermal expansion coefficient versus fiber aspect ratio.

REFERENCES

Benveniste, Y., Dvorak, G. J. and Chen, T. (1989). Stress fields in composites with coated inclusions. *Mech. Mater.* **7**, 305–317.

Chang, J. S. and Cheng, C. H. (1991). Thermoelastic properties of composites reinforced with randomly oriented short coated fibers. *Mech. Mater.* (submitted).

Chen, T., Dvorak, G. J. and Benveniste, Y. (1990). Stress fields in composites reinforced by coated cylindrically orthotropic fibers. *Mech. Mater.* **9**, 17–32.

Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion and related problems. *Proc. R. Soc.* **A241**, 376–396.

Hatta, Y. and Taya, M. (1987). Thermal stress in a coated short fiber composite. *J. Engng Mater. Tech.* **109**, 59–63.

Mikata, Y. and Taya, M. (1985). Stress field in and around a coated short fiber in an infinite matrix subjected to uniaxial and biaxial loadings. *J. Appl. Mech.* **52**, 19–24.

Mikata, Y. and Taya, M. (1986). Thermal stress in a coated short fiber composite. *J. Appl. Mech.* **53**, 681–689.

Mori, T. and Tanaka, K. (1973). Average stress in matrix and average elastic energy of materials with misfitting inclusions. *Acta Metall.* **21**, 571–574.

Mura, T. (1982). *Micromechanics of Defects in Solids*. Martinus Nijhoff, Dordrecht, The Netherlands.

Pagano, N. J. and Tandon, G. P. (1988). Elastic response of multidirectional coated-fiber composites. *Comp. Sci. Tech.* **31**, 273–293.

Pagano, N. J. and Tandon, G. P. (1990). Thermo-elastic model for multidirectional coated-fiber composites: Traction formulation. *Comp. Sci. Tech.* **38**, 247–269.

Tong, J. and Jasiuk, I. (1990). Thermal stresses and thermal expansion coefficients of composites reinforced with coated spherical particles. In *Controlled Interphases in Composite Materials* (Edited by H. Ishida), pp. 539–548. Elsevier, Amsterdam.

Walpole, L. J. (1967). The elastic field of an inclusion in an anisotropic medium. *Proc. R. Soc. London* **A300**, 270–289.

Walpole, L. J. (1978). A coated inclusion in an elastic medium. *Math. Proc. Comb. Phil. Soc.* **83**, 495–506.

APPENDIX A: VOLUME INTEGRATION OVER A THIN COATING LAYER

Consider an ellipsoid with circular cross-section and whose long axis is l and short axis is d. The equation of surface of such ellipsoidal inclusion is

$$\left(\frac{\bar{x}_1}{l}\right)^2+\left(\frac{\bar{x}_2}{d}\right)^2+\left(\frac{\bar{x}_3}{d}\right)^2=1$$

in which x_1 is the longitudinal axis. Introducing the dimensionless quantities $x_i = \bar{x}_i/d$ and $t = l/d$, the above equation can be rewritten as

$$\left(\frac{x_1}{t}\right)^2+x_2^2+x_3^2 = 1. \tag{A1}$$

Let

$$f(x) = \left(\frac{x_1}{t}\right)^2+x_2^2+x_3^2-1. \tag{A2}$$

Thus, the unit normal is

$$\mathbf{n} = (n_1, n_2, n_3) = \frac{\nabla f}{|\nabla f|}. \tag{A3}$$

Introducing polar coordinates in the x_2–x_3 section:

$$r = [x_2^2+x_3^2]^{1/2} = \left[1-\left(\frac{x_1}{t}\right)^2\right]^{1/2}; \quad \theta = \tan^{-1}\frac{x_3}{x_2}. \tag{A4}$$

Then (A1) can be rewritten as

$$\left(\frac{x_1}{t}\right)^2+r^2 = 1 \tag{A5}$$

and

$$n_1 = \frac{\dfrac{x_1}{t^2}}{\left[\left(\dfrac{x_1}{t^2}\right)^2+1-\left(\dfrac{x_1}{t}\right)^2\right]^{1/2}}, \quad n_2 = \frac{\left[1-\left(\dfrac{x_1}{t^2}\right)^2\right]^{1/2}\cos\theta}{\left[\left(\dfrac{x_1}{t^2}\right)^2+1-\left(\dfrac{x_1}{t}\right)^2\right]^{1/2}}, \quad n_3 = \frac{\left[1-\left(\dfrac{x_1}{t^2}\right)^2\right]^{1/2}\sin\theta}{\left[\left(\dfrac{x_1}{t^2}\right)^2+1-\left(\dfrac{x_1}{t}\right)^2\right]^{1/2}}. \tag{A6}$$

Volume integration over thin coating layer

Under the assumption of thin coating and hence the constant variation of stresses and strains through the thickness of the coating, the volume average of a stress or strain function, being a function of surface direction, i.e. $F(\mathbf{n})$, over the domain of coating, $\Omega_2-\Omega_1$, can be approximated by surface integration over the fiber surface according to the following formula:

$$\langle F\rangle_2 = \int_{\Omega_2-\Omega_1} F\,\mathrm{d}V \Big/ \int_{\Omega_2-\Omega_1}\mathrm{d}V \sim \int_{-t}^{t}\int_0^{2\pi} F(\mathbf{n})\,\mathrm{d}s\,r\,\mathrm{d}\theta \Big/ \int_{-t}^{t}\int_0^{2\pi}\mathrm{d}s\,r\,\mathrm{d}\theta \tag{A7}$$

in which $\mathrm{d}s^2 = \mathrm{d}x_1^2+\mathrm{d}r^2$. It can be easily shown that

$$\left(\frac{\mathrm{d}r}{\mathrm{d}x_1}\right)^2 = \frac{x_1^2}{t^4-t^2x_1^2}. \tag{A8}$$

So,

$$\mathrm{d}s = \left[1+\left(\frac{\mathrm{d}r}{\mathrm{d}x_1}\right)^2\right]^{1/2}\mathrm{d}x_1 = \left[\frac{(1-t^2)x_1^2+t^4}{t^4-t^2x_1^2}\right]^{1/2}\mathrm{d}x_1. \tag{A9}$$

Now (A7) can be written as

$$\begin{aligned}\langle F\rangle_2 &= \int_{-t}^{t}\int_0^{2\pi} F(\mathbf{n})\cdot\left[\frac{(1-t^2)x_1^2+t^4}{t^4-t^2x_1^2}\right]^{1/2}\cdot\left[1-\left(\frac{x_1}{t}\right)^2\right]^{1/2}\mathrm{d}x_1\,\mathrm{d}\theta \Big/ \int_{-t}^{t}\int_0^{2\pi}\mathrm{d}s\,r\,\mathrm{d}\theta \\ &= \int_{-t}^{t}\int_0^{2\pi} F(\mathbf{n})\cdot\left[\frac{(1-t^2)x_1^2+t^4}{t^4}\right]^{1/2}\mathrm{d}x_1\,\mathrm{d}\theta \Big/ \int_{-t}^{t}\int_0^{2\pi}\mathrm{d}s\,r\,\mathrm{d}\theta.\end{aligned} \tag{A10}$$

APPENDIX B: A SAMPLE CALCULATION FOR TYPICAL FOURTH ORDER TENSORS

In this work, a lot of fourth order tensor algebra has been involved. Some typical ones requiring an average over the domain of coating layer, such as $\mathbf{D}^3$ in eqn (21a) whose tensorial "kernel" is a function of surface direction cosine of the coating layer, are rather complicated. The algebra can however be simplified if a shorthand notation is used. Thus to facilitate the interested readers, a typical sample calculation for these tensors is shown below in detail. For simplification, the coating material is assumed to be isotropic, which in fact is rather realistic.

We start from the most general symmetric fourth-rank tensor $\mathbf{C}(C_{ijkl})$. If the symmetries $C_{ijkl} = C_{ijlk}$ are allowed, then this general tensor is given according to the following expression [see, for example, Walpole (1978)]:

$$\begin{aligned} C_{ijkl} &= \alpha(\delta_{ij}-n_in_j)(\delta_{kl}-n_kn_l)+\beta_1(\delta_{ij}-n_in_j)n_kn_l+\beta_2(\delta_{kl}-n_kn_l)n_in_j+\gamma n_in_jn_kn_l+\xi[(\delta_{ik}-n_in_k)(\delta_{jl}-n_jn_l) \\ &\quad +(\delta_{jk}-n_jn_k)(\delta_{il}-n_in_l)-(\delta_{ij}-n_in_j)(\delta_{kl}-n_kn_l)]+\eta[(\delta_{ik}-n_in_k)n_jn_l+(\delta_{il}-n_in_l)n_jn_k+(\delta_{jl}-n_jn_l)n_in_k \\ &\quad +(\delta_{jk}-n_jn_k)n_in_l] \\ &= (\alpha-\beta_1-\beta_2+\gamma-4\eta+\xi)n_in_jn_kn_l+(\alpha-\xi)\delta_{ij}\delta_{kl}+\xi(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})+(\beta_1+\xi-\alpha)n_kn_l\delta_{ij}+(\beta_2+\xi-\alpha)n_in_j\delta_{kl} \\ &\quad +(\eta-\xi)[\delta_{ik}n_jn_l+\delta_{il}n_jn_k+\delta_{jl}n_in_k+\delta_{jk}n_in_l]. \end{aligned} \tag{B1}$$

The above equation can be written in the shorthand notation

$$\mathbf{C} = (2\alpha, \beta_1, \beta_2, \gamma, 2\xi, 2\eta). \tag{B2}$$

For a transversely isotropic material,

$$\begin{aligned} \alpha &= (C_{22}+C_{23})/2, & \beta_1 &= C_{21} \\ \beta_2 &= C_{12}, & \gamma &= C_{11} \\ \xi &= C_{44} = (C_{22}-C_{23})/2, & \eta &= C_{55} = C_{66} \end{aligned} \tag{B3}$$

where the C_{ij}s are the reduced notation for $\mathbf{C}$. Consider now the two fourth-rank tensors given as

$$\mathbf{C} = (2\alpha, \beta_1, \beta_2, \gamma, 2\xi, 2\eta) \quad \mathbf{C}' = (2\alpha', \beta_1', \beta_2', \gamma', 2\xi', 2\eta'). \tag{B4}$$

Then it can be shown that

$$\mathbf{C}+\mathbf{C}' = (2\alpha+2\alpha', \beta_1+\beta_1', \beta_2+\beta_2', \gamma+\gamma', 2\xi+2\xi', 2\eta+2\eta') \tag{B5}$$

and

$$\mathbf{C}\cdot\mathbf{C}' = (4\alpha\alpha'+2\beta_1\beta_2', 2\alpha\beta_1'+\beta_1\gamma', 2\alpha'\beta_2+\beta_2'\gamma, \gamma\gamma'+2\beta_2\beta_1', 4\xi\xi', 4\eta\eta'). \tag{B6}$$

Now the fourth-rank unit tensor $\mathbf{I}$ can be expressed as

$$I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) = (1, 0, 0, 1, 1, 1). \tag{B7}$$

Let $\mathbf{C}\cdot\mathbf{C}' = \mathbf{I}$. Then it can easily be shown that

$$\mathbf{C}\cdot\mathbf{C}' = \mathbf{C}'\cdot\mathbf{C} = \mathbf{I} \tag{B8}$$

and

$$\begin{aligned} \alpha' &= \frac{\gamma}{4\alpha\gamma-4\beta_1\beta_2}, & \beta_2' &= \frac{-\beta_2}{2\alpha\gamma-2\beta_1\beta_2} \\ \gamma' &= \frac{\alpha}{2\alpha\gamma-2\beta_1\beta_2}, & \beta_1' &= \frac{-\beta_1}{2\alpha\gamma-2\beta_1\beta_2} \\ \xi' &= \frac{1}{4\xi}, & \eta' &= \frac{1}{4\eta}. \end{aligned} \tag{B9}$$

Now let

$$K^c_{ij} = C^c_{ikjl}n_kn_l = [\lambda_c\delta_{ik}\delta_{jl}+\mu_c(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk})]n_kn_l \tag{B10}$$

or

$$\mathbf{K}^{c^{-1}} = \frac{1}{\mu_c}\mathbf{I}-\frac{\lambda_c+\mu_c}{\mu_c(\lambda_c+2\mu_c)}\mathbf{nn}. \tag{B11}$$

Define $\mathbf{P}^m$ according to

$$P^m_{ijkl} = \tfrac{1}{4}(K^{m^{-1}}_{ik}n_jn_l+K^{m^{-1}}_{il}n_jn_k+K^{m^{-1}}_{jk}n_in_l+K^{m^{-1}}_{jl}n_in_k). \tag{B12}$$

For simplicity, let us consider isotropic coated material only. Then we have the following symbolic expressions:

$$\mathbf{P}^c = \left(0,0,0,\frac{1}{\lambda_c+2\mu_c},0,\frac{1}{2\mu_c}\right) \quad \mathbf{P}^m = \left(0,0,0,\frac{1}{\lambda_m+2\mu_m},0,\frac{1}{2\mu_m}\right)$$

$$\mathbf{C}^c = (2\lambda_c+2\mu_c,\lambda_c,\lambda_c,\lambda_c+2\mu_c,2\mu_c,2\mu_c)$$

$$\mathbf{C}^c\cdot\mathbf{P}^c = \left(0,\frac{\lambda_c}{\lambda_c+2\mu_c},0,1,0,1\right) \quad \mathbf{C}^m\cdot\mathbf{P}^m = \left(0,\frac{\lambda_m}{\lambda_m+2\mu_m},0,1,0,1\right)$$

$$\mathbf{C}^c\cdot\mathbf{P}^c\cdot\mathbf{C}^c = \left(\frac{2\lambda_c^2}{\lambda_c+2\mu_c},\lambda_c,\lambda_c,\lambda_c+2\mu_c,0,2\mu_c\right)$$

$$\mathbf{C}^c\cdot\mathbf{P}^c\cdot\mathbf{C}^m = \left(\frac{2\lambda_c\lambda_m}{\lambda_c+2\mu_c},\frac{\lambda_c(\lambda_m+2\mu_m)}{\lambda_c+2\mu_c},\lambda_m,\lambda_m+2\mu_m,0,2\mu_m\right). \tag{B13}$$

Thus

$$\begin{aligned}\mathbf{D}^1 &= \mathbf{C}^c-\mathbf{C}^c\cdot\mathbf{P}^c\cdot\mathbf{C}^c+\mathbf{C}^c\cdot\mathbf{P}^c\cdot\mathbf{C}^m \\ &= \left(\frac{4\mu_c^2+6\mu_c\lambda_c}{\lambda_c+2\mu_c}+\frac{2\lambda_c\lambda_m}{\lambda_c+2\mu_c},\frac{\lambda_c(\lambda_m+2\mu_m)}{\lambda_c+2\mu_c},\lambda_m,\lambda_m+2\mu_m,2\mu_c,2\mu_m\right) \\ &\equiv (2d_{11},d_{12},d_{13},d_{14},2d_{15},2d_{16})\end{aligned} \tag{B14}$$

$$\begin{aligned}\mathbf{D}^2 &= \mathbf{C}^c\cdot\mathbf{P}^c\cdot\mathbf{C}^m \\ &= \left(\frac{2\lambda_c\lambda_m}{\lambda_c+2\mu_c},\frac{\lambda_c(\lambda_m+2\mu_m)}{\lambda_c+2\mu_c},\lambda_m,\lambda_m+2\mu_m,0,2\mu_m\right) \\ &\equiv (2d_{21},d_{22},d_{23},d_{24},2d_{25},2d_{26}).\end{aligned} \tag{B15}$$

Due to the symmetry of $\mathbf{C}^m$, eqn (8) can be easily shown as

$$\begin{aligned}\mathbf{S}^0 &= \mathbf{S}-\mathbf{C}^m\cdot\mathbf{P}^m \\ &= (S_{22}+S_{23},S_{21},S_{12},S_{11},S_{22}-S_{23},2S_{66})-\left(0,\frac{\lambda_m}{\lambda_m+2\mu_m},0,1,0,1\right) \\ &= (2S_1^0,S_2^0,S_3^0,S_4^0,2S_5^0,2S_6^0)\end{aligned} \tag{B16}$$

where

$$\begin{aligned}S_1^0 &= \frac{S_{22}+S_{23}}{2}, & S_2^0 &= S_{21}-\frac{\lambda_m}{\lambda_m+2\mu_m} \\ S_3^0 &= S_{12}, & S_4^0 &= S_{11}-1 \\ S_5^0 &= \frac{S_{22}-S_{23}}{2}, & S_6^0 &= S_{66}-\tfrac{1}{2}.\end{aligned} \tag{B17}$$

We now have

$$\mathbf{C}^{c\,-1} = \left(\frac{2(\lambda_c+2\mu_c)}{4(2\mu_c^2+3\mu_c\lambda_c)},\frac{-\lambda_c}{2(2\mu_c^2+3\mu_c\lambda_c)},\frac{-\lambda_c}{2(2\mu_c^2+3\mu_c\lambda_c)},\frac{\lambda_c+\mu_c}{2(2\mu_c^2+3\mu_c\lambda_c)},\frac{1}{2\mu_c},\frac{1}{2\mu_c}\right) \tag{B18}$$

$$\mathbf{C}^{m\,-1} = \left(\frac{2(\lambda_m+2\mu_m)}{4(2\mu_m^2+3\mu_m\lambda_m)},\frac{-\lambda_m}{2(2\mu_m^2+3\mu_m\lambda_m)},\frac{-\lambda_m}{2(2\mu_m^2+3\mu_m\lambda_m)},\frac{\lambda_m+\mu_m}{2(2\mu_m^2+3\mu_m\lambda_m)},\frac{1}{2\mu_m},\frac{1}{2\mu_m}\right). \tag{B19}$$

Denoting

$$\mathbf{C}^{c\,-1}-\mathbf{C}^{m\,-1} = (2\Delta C_1,\Delta C_2,\Delta C_3,\Delta C_4,2\Delta C_5,2\Delta C_6). \tag{B20}$$

Thus the following expressions can be readily obtained

$$\begin{aligned}\mathbf{S}^0\cdot(\mathbf{C}_c^{-1}-\mathbf{C}_m^{-1}) = (&2(2S_1^0\Delta C_1+S_2^0\Delta C_3), \\ &2S_1^0\Delta C_2+S_2^0\Delta C_4,2S_3^0\Delta C_1+S_4^0\Delta C_3,2S_3^0\Delta C_2+S_4^0\Delta C_4, \\ &4S_5^0\Delta C_5,4S_6^0\Delta C_6) \equiv (2b_1,b_2,b_3,b_4,2b_5,2b_6)\end{aligned} \tag{B21}$$

$$\mathbf{S}^0\cdot(\mathbf{C}_c^{-1}-\mathbf{C}_m^{-1})\cdot\mathbf{D}^1 = (2(2b_1d_{11}+b_2d_{13}),2b_1d_{12}+b_2d_{14},2b_3d_{11}+b_4d_{13},2b_3d_{12}+b_4d_{14},4b_5d_{15},4b_6d_{16}) \tag{B22}$$

$$\mathbf{S}^0\cdot(\mathbf{C}_c^{-1}-\mathbf{C}_m^{-1})\cdot\mathbf{D}^2 = (2(2b_1d_{21}+b_2d_{23}),2b_1d_{22}+b_2d_{24},2b_3d_{21}+b_4d_{23},2b_3d_{22}+b_4d_{24},4b_5d_{25},4b_6d_{26}). \tag{B23}$$

Now let

$$\mathbf{F} \equiv \mathbf{S}^{0} \cdot (\mathbf{C}^{c^{-1}} - \mathbf{C}^{m^{-1}}) \cdot \mathbf{D}^{1} = (2d_{31}, d_{32}, d_{33}, d_{34}, 2d_{35}, 2d_{36}) \tag{B24}$$

or

$$\begin{aligned} F_{ijkl} = {} & (d_{31} - d_{32} - d_{33} + d_{34} - 4d_{36} + d_{35}) n_i n_j n_k n_l \\ & + (d_{31} - d_{35}) \delta_{ij} \delta_{kl} + d_{35} (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) \\ & + (d_{32} + d_{35} - d_{31}) \delta_{ij} n_k n_l + (d_{33} + d_{35} - d_{31}) \delta_{kl} n_i n_j \\ & + (d_{36} - d_{35}) (\delta_{ik} n_j n_l + \delta_{il} n_j n_k + \delta_{jl} n_i n_k + \delta_{jk} n_i n_l). \end{aligned} \tag{B25}$$

Now since

$$\mathbf{D}^{1} = f_{12} \langle \mathbf{F} \rangle_2 \tag{B26}$$

the component D^{1}_{ijkl} can then be computed, according to Appendix A, as

$$\begin{aligned} D^{1}_{ijkl} &= f_{12} \langle F_{ijkl} \rangle_2 \\ &= f_{12} \int_{-t}^{t} \int_{0}^{2\pi} F_{ijkl}(\mathbf{n}) \cdot \left[\frac{(1-t^2)x_1^2 + t^4}{t^4 - t^2 x_1^2} \right]^{1/2} \cdot \left[1 - \left(\frac{x_1}{t} \right)^2 \right]^{1/2} \mathrm{d}x_1 \, \mathrm{d}\theta \Big/ \int_{-t}^{t} \int_{0}^{2\pi} \mathrm{d}s \, r \, \mathrm{d}\theta \\ &= f_{12} \int_{-t}^{t} \int_{0}^{2\pi} F_{ijkl}(\mathbf{n}) \cdot \left[\frac{(1-t^2)x_1^2 + t^4}{t^4} \right]^{1/2} \mathrm{d}x_1 \, \mathrm{d}\theta \Big/ \int_{-t}^{t} \int_{0}^{2\pi} \mathrm{d}s \, r \, \mathrm{d}\theta. \end{aligned} \tag{B27}$$